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Temperature Dynamics of the Locally Perturbed Classical Ideal Gas

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In the classical gas any two particles interact with the repulsive potential iff they both are situated in the fixed region Λ. Outside Λ they move freely. We prove that this dynamical system with respect to the Gibbs measure is metrically isomorphic to the classical ideal gas.

KEY WORDS: Temperature dynamics; classical gas; dynamical system; metric isomorphism; Gibbs measure; local perturbations.

1. INTRODUCTION

This paper is the second in the series (the first is Ref. 1) where we prove asymptotic completeness for some infinite particle systems. These systems are characterized by the following property: some region $\Lambda \subset \mathbb{R}^\nu$ is fixed and any two particles interact iff they are both situated in Λ. Any particle outside Λ moves as in the ideal gas. Such systems were considered earlier for the stochastic dynamics[(2)] and from the point of view of the kinetic equations.[(3)]

Here we consider the classical gas with the repulsive local potential

$$\sum_{i,j} \chi_\Lambda(x_i)\, \chi_\Lambda(x_j)\, \Phi(|x_i - x_j|)$$

Our main result is Theorem 6.1 where we prove that (for any temperature) it is metrically isomorphic to the classical ideal gas. The methods here are quite different from the methods of Ref. 1.

In Sections 2 and 3 we give the exact definitions of the both dynamical systems. In Section 4 we give the main probabilistic and geometrical con-

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struction. It is reminiscent of Sinai's cluster dynamics[7]: during the small time interval there are finite groups of particles and these groups do not interact with one another. In Section 5 we define direct and inverse Möller morphisms on the phase space in the way similar to classical finite particle systems.[4–6] The main result is the Theorem 6.1 proven in Section 6.

We think that results of this paper continue to hold under much more general condition on potential in spite of complexities which appear while observing.

2. THE CLASSICAL IDEAL GAS

We only fix notations here (see Ref. 6).

We define the Borel measure

$$d\rho = \text{const} \exp[-\beta(v, v)]\, dv\, dx$$

on the phase space $\mathbb{R}^{2\nu} = \mathbb{R}^\nu_x \times \mathbb{R}^\nu_v$ of the classical particle, where $(\cdot, \cdot)$ is the standard scalar product in $\mathbb{R}^\nu_v$, $dv = dv^1 \cdots dv^\nu$, $dx = dx^1 \cdots dx^\nu$.

The phase space Ω of the classical gas is the set of all locally finite configurations $\omega \subset \mathbb{R}^{2\nu}$. Ω is a Polish space with the well-known topology.[6]

We define the integer-valued function on Ω

$$\kappa_B(\omega) = \text{card}(\omega \cap B)$$

for any bounded set $B \subset \mathbb{R}^{2\nu}$. The σ algebra $\tilde{\Sigma}$ in Ω is defined to be the minimal σ algebra such that all functions $\kappa_B(\omega)$ are measurable with respect to it. We define

$$C_{B,k} = \{\omega \in \Omega \colon \kappa_B(\omega) = k\}, \qquad k = 0, 1, 2,...$$

The Poisson measure μ on $(\Omega, \tilde{\Sigma})$ such that

$$\mu(C_{B,k}) = [\rho(B)]^k \exp[-\rho(B)]/k! \tag{2.1}$$

is defined in the usual way. We define σ algebra Σ as the completion of $\tilde{\Sigma}$ with respect to μ.

Time evolution T^t of the ideal gas is defined by

$$T^t\omega = \{S^t(x_i, v_i)\} \qquad \text{if} \quad \omega = \{(x_i, v_i)\}$$

and

$$S^t(x, v) = (x + vt, v)$$

So $(\Omega, \Sigma, \mu, T^t)$ is the classical dynamical system (ideal gas).

3. THE LOCAL PERTURBATION OF THE CLASSICAL IDEAL GAS

Let us consider the infinitely differentiable function $\Phi(r)$ (repulsive potential) on the interval $(0, +\infty)$ with the following properties:

$$\begin{array}{lll} (1) & \Phi(r) & \text{is monotone decreasing} \\ (2) & \Phi(r) > 0, & \text{i.e., } \Phi(r) \text{ is strictly positive} \\ (3) & \Phi(r) \to +\infty & \text{as } r \to +0 \end{array} \tag{3.1}$$

Then

$$\operatorname{grad} \Phi(|x|) = -\varphi(|x|)x$$

where $\varphi(r)$ is an infinitly differentiable positive function on the interval $(0, +\infty)$. We shall suppose that there exist positive constants A, C, and d such that for $r \in (0, A)$

$$\varphi(r) > Cr^{-d} \tag{3.2}$$

d will be defined later on.

Let Λ be bounded open convex domain in $\mathbb{R}_x^\nu$ with sufficiently smooth boundary. We define the local potential $\Phi_\Lambda(\omega)$ as the function

$$\Phi_\Lambda(\omega) = \sum_{i,j} \chi_\Lambda(x_i)\, \chi_\Lambda(x_j)\, \Phi(|x_i - x_j|) \tag{3.3}$$

on (Ω, Σ) where χ_Λ is the indicator of Λ.

The local potential Φ_Λ corresponds to locally perturbed dynamics T_Λ^t which acts on any ω from some subset Ω_1 (the definition of Ω_1 will be given later on). We shall prove that $\mu(\Omega_1) = 1$. We shall also prove that for every $\omega \in \Omega_1$ there exists an infinite sequence of times $t_1(\omega) < t_2(\Omega) < \cdots$ tending to the infinity such that in every $t_i(\omega)$ exactly one particle of the configuration $T_\Lambda^{t_i(\omega)}\omega$ hits $\partial\Lambda$ from the outer region Λ^c and there are no particles which hit $\partial\Lambda$ from Λ^c in any other moment. So T_Λ^t has the following properties:

(1) Particles situated outside Λ move freely.

(2) If the particle hitting $\partial\Lambda$ from Λ^c in the moment t_i has the velocity v until the moment t_i, then its velocity changes according to the following rule: let x_0 be the point of $\partial\Lambda$ which the particle hits at the moment t_i and $v^\perp$, the orthogonal component of v with respect to $\partial\Lambda$

at the point x_0. Let one have exactly n particles inside Λ at the moment t_i with positions $x_1, x_2, \ldots, x_n$. We denote

$$U_{x_0} = \sum_{i=1}^{n} \Phi(|x_0 - x_i|)$$

(a) In the case $(1/2)(v^{\perp})^2 > U_{x_0}$ we shall assume that our particle has the orthogonal component of the velocity $v^{\perp}(t_i)$ at the moment t_i such that

$$(1/2)\{(v^{\perp})^2 - [v^{\perp}(t_i)^2]\} = U_{x_0}$$

The tangential component of the velocity is not changed.

(b) If we have $(1/2)(v^{\perp})^2 = 0$, i.e., the particle moves along some tangent line to $\partial\Lambda$ then it continues its free moving.

(c) In the case when $0 < (1/2)(v^{\perp})^2 < U_{x_0}$ one has the case of the elastic reflection, i.e.,

$$v^{\perp}(t_i) = -v^{\perp}$$

and the tangential component of its velocity is preserved.

The movement of particles inside Λ in the interval (t_i, t_{i+1}) takes place in accordance with the Hamiltonian dynamics. Moreover if the particle hits $\partial\Lambda$ from within (we also suppose that not more than one particle can hit $\partial\Lambda$ from within at any given moment, see below) then the velocity has the jump similar to the case (a) with minor change that the jump occurs after the moment of hitting. So the total energy of the system is conserved in this case also.

So $T_\Lambda^t \omega$ is defined for any ω and all t until the moment $t_0(\omega)$ when two or more particles will be situated on $\partial\Lambda$. We shall prove later that the probability of $t_0(\omega) = \infty$ is equal to 1.

Definition of Ω_1 (the domain of definition of T_Λ^t). Ω_1 consists of all ω such that (1) the projection of $T^t\omega$ onto $\mathbb{R}_x^\nu$ is locally finite for any t; (2) there exists an infinite sequence of moments $t_1^*(\omega) < t_2^*(\omega) < \cdots$ tending to the infinity and such that for any $t_i^*(\omega)$

$$|T_\Lambda^{t_i^*(\omega)} \omega \cap \partial\Lambda| = 1$$

and for any $t \neq t_i^*(\omega)$, $i = 1, 2, \ldots$

$$T_\Lambda^t \omega \cap \partial\Lambda = \varnothing$$

It is easy to see that Ω_1 is invariant with respect to T_Λ^t.

Proposition 3.1.

$$\mu(\Omega_1) = 1$$

Proof. If $\omega \bar{\in} \Omega_1$ then there exists $t = t(\omega)$ such that at least two particles hit $\partial\Lambda$ at the moment t. The following cases can occur:

(a) Two particles hit $\partial\Lambda$ from Λ^c at the moment t. This case has zero μ measure due to the known properties of the free dynamics.

(b) Two particles hit $\partial\Lambda$ from within. This case also has zero μ measure due to the properties of Liouville dynamics of finite number of particles.

(c) At least one particle from inside Λ and at least one particle from Λ^c hit $\partial\Lambda$ at the moment t. Then we proceed as in the case (b) choosing some sequence $\Lambda_k \nearrow \mathbb{R}_x^\nu$ and using the properties of Liouville dynamics for finite number particles in any Λ_k. Then ω belongs to the denumerable union of sets of zero measure.

Let us define the Gibbs measure μ^Φ by

$$\frac{d\mu^\Phi}{d\mu} = Z^{-1} \exp[-\beta\Phi_\Lambda(\omega)] \tag{3.4}$$

where

$$Z = \int_\Omega \exp[-\beta\Phi_\Lambda(\omega)]\, d\mu \quad \blacksquare$$

Proposition 3.2. μ and μ^Φ are absolutely continuous with respect to each other. So

$$\mu^\Phi(\Omega_1) = 1$$

Proof. This is quite evident. $\blacksquare$

To prove the T_Λ^t invariance of μ^Φ we shall define the smooth approximation to the dynamics T_Λ^t. Let $f(x)$ be a C^∞ function which is equal to 0 for $x < -1$ and equal to 1 for $x > 0$. It is assumed also to be monotone increasing on $(-1, 0)$. We define the family of functions

$$f_\delta(x) = f\left(\frac{x}{\delta}\right), \qquad \delta > 0 \tag{3.5}$$

If $\nu = 1$ and $\Lambda = (0, a)$ then we define the smoothed indicator by the formula

$$\chi_\Lambda^\delta(x)=\begin{cases}0, & x<-\delta\\ f_\delta(x), & -\delta<x<0\\ 1, & 0<x<a\\ f_\delta(a-x), & a<x<a+\delta\\ 0, & a+\delta<x\end{cases}$$

If $v>1$ then we choose

$$\chi_\Lambda^\delta(x)=\begin{cases}1, & x\in\Lambda\\ f_\delta(-r), & x\bar{\in}\Lambda\end{cases}$$

where r is the distance $r=r(x,\partial\Lambda)$.

Then we define the smoothed potential

$$\Phi_\Lambda^\delta(\omega)=\sum_{i,j}\chi_\Lambda^\delta(x_i)\,\chi_\Lambda^\delta(x_j)\,\Phi(|x_i-x_j|)$$

This potential defines the new perturbed dynamics T_δ^t of the classical gas.

Lemma 3.1. For any t_0 and any $\omega\in\Omega_1$,

$$T_\delta^{t_0}\omega\to T_\Lambda^{t_0}\omega \qquad \text{if} \quad \delta\to 0$$

in the standard topology (see Section 2).

Proof. As $\omega\in\Omega_1$ then there exists only a finite number of particles $(x_1,v_1),\ldots,(x_n,v_n)$ which enter $\bar{\Lambda}$ during the time interval $[0,t_0]$ or are situated there at the moment $t=0$. So there exists $\delta_0>0$ such that any dynamics T_δ^t, $\delta<\delta_0$, acts on any other particles as the free dynamics. Let $0\leqslant t_1<t_2<\cdots<t_k\leqslant t_0$ be the sequence of all moments of time when exactly one particle hits $\partial\Lambda$ under the dynamics T_Λ^t.

We shall prove that

$$T_\delta^t\omega\to T_\Lambda^t\omega$$

first for $t<t_1$, then for $t\in[t_1,t_2]$ and so on by induction. For $t<t_1$ this is evident. There exists $\varepsilon>0$ such that for sufficiently small $\delta_0>0$ and any $\delta<\delta_0$ only one particle [say, (x_1,v_1) and it is situated outside Λ] can be situated in the domain $\Lambda_\delta\backslash\Lambda$ during the time interval $(t_1-\varepsilon,t_1+\varepsilon)$, where $\Lambda_\delta=\operatorname{supp}\chi_\Lambda^\delta$. As the energy of this particle is bounded uniformly in δ and in $t\in(t_1-\varepsilon,t_1+\varepsilon)$ then the time when this particle spends in $\Lambda_\delta\backslash\Lambda$ is $O(\delta)$, its velocity is also uniformly bounded and so its $x_1^\delta(t_1)$ under the dynamics T_δ^t tends to $x_1(t_1)$. Moreover tangential component of the strength acting on this particle is also uniformly bounded (this follows from the choice of χ_Λ^δ). So the tangential component of the velocity of this particle also tends to that which takes place under the dynamics T_Λ^t. The law of the conservation of energy completes the proof of our assertion for the first particle. Then we proceed by induction and using the continuous dependence of the dynamics from the initial conditions.

There exist Gibbs measure μ^δ corresponding to potentials Φ^δ_Λ:

$$\frac{d\mu^\delta}{d\mu} = Z_\delta^{-1} \exp[-\beta\Phi^\delta_\Lambda(\omega)]$$

where

$$Z_\delta = \int_\Omega \exp[-\beta\Phi^\delta_\Lambda(\omega)]\, d\mu$$

As all potentials Φ^δ_Λ are infinitely differentiable then μ^δ are invariant with respect to T^t_δ.

Proposition 3.3. The Gibbs measure μ^Φ is invariant with respect to the dynamics T^t_Λ.

Proof. Let f be any continuous, bounded function on (Ω, Σ). We need to prove that

$$\int_\Omega f(\omega)\, d\mu^\Phi = \int_\Omega f(T^t_\Lambda \omega)\, d\mu^\Phi$$

for any t. As f is bounded then by the Lebesque theorem for any $\varepsilon > 0$ there exists δ'_0 such that for any $\delta < \delta'_0$

$$\left| \int_{\Omega_1} f(T^t_\delta \omega)\, d\mu^\Phi - \int_{\Omega_1} f(T^t_\Lambda \omega)\, d\mu^\Phi \right| < \varepsilon$$

But by the weak convergence of μ^δ to μ^Φ there exists δ''_0 such that for any $\delta < \delta''_0$

$$\left| \int_\Omega f(\omega)\, d\mu^\delta - \int_\Omega f(\omega)\, d\mu^\Phi \right| < \varepsilon$$

So if $\delta_0 = \min(\delta'_0, \delta''_0)$ we get

$$\left| \int_\Omega f(\omega)\, d\mu^\Phi - \int_\Omega f(T^t_\Lambda \omega)\, d\mu^\Phi \right|$$

$$\leqslant \left| \int_\Omega f(\omega)\, d\mu^\Phi - \int_\Omega f(T^t_\delta \omega)\, d\mu^\Phi \right| + \varepsilon$$

$$\leqslant \int_\Omega f(\omega)\, d\mu^\delta - \int_\Omega f(T^t_\delta \omega)\, d\mu^\delta \Bigg| + 3\varepsilon = 3\varepsilon$$

The proposition is proved. ∎

Definition. $(\Omega_1, \Sigma, \mu^{\Phi}, T^t_{\Lambda})$ is called the locally perturbed classical gas.

4. CLUSTER PROPERTIES OF THE PERTURBED DYNAMICS

Lemma 4.1. Let

$$d > 2(\nu + 1) \tag{4.1}$$

where d is the constant from (3.2). Then for any $\varepsilon > 0$ there exists $T = T(\varepsilon, d, \nu, \Lambda)$ such that the following condition holds: let any configuration of n particles with different coordinates in Λ at the moment t be given. Then all these particles except at most one with its velocity not exceeding ε will leave Λ in the time interval $(t, t+T)$ if no particles enter Λ during this time. We stress that T does not depend on n and the initial configuration of particles inside Λ.

Proof. We use some ideas of Ref. 4. Let n particles (x_i, v_i), $i \in I$, $|I| = n$, be in Λ at the moment $t = 0$.

Let us put $x_i = x_i(t)$ and

$$P(t) = \frac{1}{2} \sum_{i,j \in I} r_{ij}^2, \qquad r_{ij} = x_i - x_j = -r_{ji}$$

Then

$$\begin{aligned} \dot{P}(t) &= \sum_{i,j \in I} \dot{r}_{ij} r_{ij} \\ \ddot{P}(t) &= \sum_{i,j} \ddot{r}_{ij} r_{ij} + \sum_{i,j} \dot{r}_{ij}^2 \end{aligned} \tag{4.2}$$

(we consider only the moments t such that all particles are still in Λ). As

$$\ddot{x}_i = -\sum_{j \in I} \operatorname{grad} \Phi(|x_i - x_j|) = \sum_{j \in I} \varphi(|r_{ij}|)\, r_{ij}$$

then

$$\ddot{r}_{ij} = \sum_s \varphi(|r_{is}|)\, r_{is} - \sum_s \varphi(|r_{sj}|)\, r_{js} \tag{4.3}$$

We can rewrite the first sum in (4.2) using (4.3)

$$\sum_{i,j \in I} \ddot{r}_{ij} r_{ij} = \sum_{i,j} r_{ij} \left[\sum_s \varphi(|r_{is}|)\, r_{is} + \sum_s \varphi(|r_{js}|)\, r_{sj} \right]$$
$$= \sum_{i,j} \varphi(|r_{ij}|)\, r_{ij} \left(\sum_s r_{is} + \sum_s r_{sj} \right)$$
$$= \sum_{i,j} \varphi(|r_{ij}|)\, r_{ij} \sum_s (x_i - x_s + x_s - x_j)$$
$$= n \sum_{i,j} \varphi(|r_{ij}|)\, r_{ij}^2$$

and

$$\ddot{P} = n \sum_{i,j} \varphi(|r_{ij}|)\, r_{ij}^2 + \sum_{i,j} \dot{r}_{ij}^2 > 0 \tag{4.4}$$

So $P(t)$ is an unbounded convex function. Let us put

$$R \stackrel{\text{def}}{\equiv} \sup_{x_i \in \Lambda, i \in I} \frac{1}{2} \sum_{i,j} (x_i - x_j)^2 = \sup P \leqslant n^2 C_1(\Lambda) \tag{4.5}$$

where $C_1(\Lambda)$ does not depend on n,

$$a = \inf_{x_i \in \Lambda, i \in I} \left(n \sum_{i,j} \varphi(|r_{ij}|)\, r_{ij}^2 \right) \geqslant C_2(\Lambda)\, n^{2 + (d-2)/v} \tag{4.6}$$

where $C_2(\Lambda)$ also does not depend on n, this estimation will be proved in Lemma 4.2. Let $C = \max(\operatorname{diam} \Lambda, 1)$. Then for $\dot{P}(0) \geqslant C^2 n^2$ it follows from (4.4) that

$$\ddot{P}(0) \geqslant n^{-2} \dot{P}(0) + a$$

So when n is sufficiently large and $d > 2(v+1)$ we have $\dot{P}(t_n) \geqslant 0$ if $t_n = C^2 n^{-1-\alpha}$, where $0 < \alpha < d - 2(v+1)$.

Now we must solve the inequality

$$P(t) \geqslant R$$

with the respect to t. If T_n satisfies this inequality then all n particles can not stay in Λ during the time interval $[0, T_n]$. So by (4.1), (4.5), and (4.6)

$$T_n \leqslant t_n + (2R/a)^{1/2} \leqslant C(\Lambda)\, n^{-1-\alpha}$$

Using this estimation we have

$$T(n) \leqslant \sum_{k=2}^{n} T_k \leqslant \sum_{k=2}^{\infty} T_k < +\infty$$

where $T(n)$ is the first exit time of $n-1$ particles from Λ. Then we can put

$$T = \sum_{k=2}^{\infty} T_k + C/\varepsilon$$

and Lemma 4.1 is proved. ▮

Lemma 4.2.

$$a > C_2(\Lambda)\, n^{2+(d-2)/v}$$

where $C_2(\Lambda)$ does not depend on n.

Proof. We can divide Λ into $\frac{1}{2}n$ domains, each having diameter not exceeding $Cn^{1/v}$. At least $n/2$ pairs of particles will be in adjacent domains. So

$$a > n \inf\left(\sum r_{ij}^{2-d}\right) > Cn\, n/2\, n^{(d-2)/v}$$

$$= C_2(\Lambda)\, n^{2+(d-2)/v} \quad ▮$$

Lemma 4.3. Under the condition (4.1) for almost all $\omega \in \Omega_1$ (with respect to μ or μ^{Φ}) there exists an infinite sequence of moments of time $t_1 < t_2 < \cdots$, $t_i \to +\infty$, $i \to +\infty$ such that

$$T_\Lambda^{t_i}\omega \cap \bar{\Lambda} = \varnothing$$

and so on any particle spends only finite time in Λ.

Proof. We shall prove that for any $t_0 > 0$ there exist $t_0 < s_1 < s_2 < t_1 < +\infty$ such that (1) $s_2 - s_1 > T$ and there is no particle entering Λ during the time interval $[s_1, s_2]$;

(2) in the time interval $[s_2, t_1]$ exactly one particle enters $\bar{\Lambda}$;

(3) there are no particles in Λ at the moment t_1.

It is easy to prove that there exists an infinite number of time intervals $s_1^{(i)} < s_2^{(i)} < t_1^{(i)} < s_1^{(i+1)} < \cdots$ such that

(A_i) $s_2^{(i)} - s_1^{(i)} > T$ and no particles enter Λ in the time interval $[s_1^{(i)}, s_2^{(i)}]$;

(B_i) $t_1^{(i)} - s_2^{(i)} > T$ and exactly one particle enters Λ in the time interval $[s_2^{(i)}, t_1^{(i)}]$.

From the properties of the Poisson random field it follows that (A_i) and (B_i) are independent events. We want to prove that there exists number i such that there are no particles in Λ at moment $t_1^{(i)}$.

We know from Lemma 4.1 that at the moment $s_2^{(i)}$ there is at most one particle in $\bar{\Lambda}$. Let one have exactly one particle in Λ (the case when there is no particle is evident). We know that its velocity does not exceed ε. Simple geometrical consideration shows that the probability that both particles leave Λ before $t_1^{(i)}$ is bounded from below by some $\delta > 0$ (δ is independent of the coordinate and the velocity of the particle in Λ at the moment $s_2^{(i)}$). The proof is concluded then by the Borel–Cantelli lemma. ∎

5. MÖLLER MORPHISMS ON THE PHASE SPACE

Let us define $\Omega_2 \in \Sigma$ to be the subset of all $\omega \in \Omega_1$ for which there exists an infinite sequence of moments $w_1(\omega) < w_2(\omega) < \cdots$ tending to the infinity such that

$$T_\Lambda^{w_i(\omega)}\omega \cap \bar{\Lambda} = \varnothing \tag{5.1}$$

for all i. Ω_2 is evidently invariant with respect to T_Λ^t.

Lemma 5.1:

$$\mu(\Omega_2) = \mu^\Phi(\Omega_2) = 1 \tag{5.2}$$

and for any $\omega \in \Omega_2$ there exist

$$\gamma_\pm \omega = \lim_{t \to \pm\infty} T^{-t} T_\Lambda^t \omega \tag{5.3}$$

$\gamma_\pm$ are called the direct Möller morphisms.

Proof. (5.2) is proved in Lemma 4.3.

(5.3) follows readily from (5.1) as for any particle $(x_i, v_i) = (\omega)_i \in \omega$ there exists $t_j < +\infty$ such that for any $t > t_j$

$$(T_\Lambda^t \omega)_i = (T_\Lambda^{t-t_j} T_\Lambda^{t_j}\omega)_i = (T^{t-t_j} T_\Lambda^{t_j}\omega)_i$$

and so for all $t > t_j$

$$(T^{-t} T_\Lambda^t \omega)_i = (T^{-t_j} T_\Lambda^{t_j}\omega)_i = (\gamma_\pm \omega)_i \tag{5.4}$$

Now we shall define the inverse Möller morphisms

$$\bar{\gamma}_\pm \omega = \lim_{t \to \pm\infty} T_\Lambda^{-t} T^t \omega \tag{5.5}$$

using the "dual" construction.

Let us fix $\omega \in \Omega_1$ and consider the dynamics $T^t\omega$. Some particles $(\omega)_i = (x_i, v_i) \in \omega$ can enter Λ. We shall denote $0 \leqslant u_1(\omega) \leqslant u_2(\omega) \leqslant \cdots$ the moments of their exit out of Λ. We note that the particle $(\omega)_i$ goes out from Λ at the moment $u_i \in [0, \hat{t}]$ iff the particle $(T^{\hat{t}}\omega)_i$ of the configuration $T^{\hat{t}}\omega$ enters $\bar{\Lambda}$ at the moment $t - u_i \in [0, \hat{t}]$ under the dynamics T_Λ^{-t}.

As in Section 4 one can prove that there exists an infinite number of time intervals $t_1^{(i)} < s_2^{(i)} < s_1^{(i)} < t_1^{(i+1)} < \cdots$ such that

(A_i') $s_1^{(i)} - s_2^{(i)} > T$ and there is no $u_j(\omega) \in [s_2^{(i)}, s_1^{(i)}]$;
(B_i') $s_2^{(i)} - t_1^{(i)} > T$ and there is exactly one $u_k(\omega) \in [t_1^{(i)}, s_2^{(i)}]$.

Let us define Ω_2' as the subset of all $\omega \in \Omega_1$ such that (A_i') and (B_i') are fulfilled. So quite similar to Lemma 4.3 and Lemma 5.1 one can prove the following.

Lemma 5.1′:

$$\mu(\Omega_2') = \mu^{\Phi}(\Omega_2') = 1 \tag{5.6}$$

and for any $\omega \in \Omega_2'$ there exist the inverse Möller morphisms

$$\bar{\gamma}_\pm \omega = \lim_{t \to \pm\infty} T_\Lambda^{-t} T^t \omega \tag{5.7}$$

In particular for any $\omega \in \Omega_2'$ there exists the sequence of moments of time $t_1 < t_2 < \cdots$ tending to infinity such that

$$T_\Lambda^{-(t-t_i)} T^t \omega \cap \bar{\Lambda} = \varnothing \tag{5.8}$$

for all $t > \bar{t}(i)$, $\bar{t}(i) \to +\infty$ if $i \to \infty$.

Lemma 5.2:

$$\bar{\gamma}_\pm \Omega_2' \subseteq \Omega_2 \tag{5.9}$$

Proof. We have for $\bar{\omega} = \bar{\gamma}_+ \omega$

$$\begin{aligned} \Lambda \cap T_\Lambda^{t_i} \bar{\omega} &= \Lambda \cap \lim_{t \to +\infty} T_\Lambda^{t_i} T_\Lambda^{-t} T^t \omega \\ &= \Lambda \cap \lim_{t \to +\infty} T_\Lambda^{-(t-t_i)} T^t \omega = \Lambda \cap T_\Lambda^{-(t-t_i)} T^t \omega = \varnothing \end{aligned}$$

Theorem 5.1. There exist direct Möller morphisms $\gamma_\pm$ on Ω_2, they are invertible on $\Omega_\pm = \operatorname{Im} \gamma_\pm(\Omega_2)$ and $\mu(\Omega_\pm) = \mu^{\Phi}(\Omega_\pm) = 1$.

Proof. The existence of direct Möller morphisms $\gamma_\pm$ on Ω_2 was proved in Lemma 5.1. Now we note that $\gamma_\pm$ is one-to-one as for any $\omega \in \Omega_2$ and any i there exists t_i such that

$$(\gamma_\pm \omega)_i = (\gamma^t \omega)_i \overset{\text{def}}{=} (T^{-t} T^t_\Lambda \omega)_i$$

if $t > t_i$ and γ^t is invertible for any t.

As $\gamma_\pm \bar{\gamma}_\pm = 1$ on Ω'_2 then the last assertion of the theorem follows from Lemmas 5.1′ and 5.2. ∎

6. METRIC ISOMORPHISM OF THE TWO DYNAMICS

Here we shall prove that γ_+^{-1} which is defined on Ω_+ is the metric isomorphism of dynamical systems $(\Omega, \Sigma, \mu, T^t)$ and $(\Omega, \Sigma, \mu^\Phi, T^t_\Lambda)$.

Lemma 6.1. For any $\omega \in \Omega_2$, $\bar{\omega} \in \Omega_+$ and any t

$$\gamma_+ T^t_\Lambda \omega = T^t \gamma_+ \omega \tag{6.1}$$

and

$$T^t_\Lambda \gamma_+^{-1} \bar{\omega} = \gamma_+^{-1} T^t \bar{\omega} \tag{6.2}$$

Proof. We shall prove (6.1). As Ω_2 is invariant with respect to T^t_Λ then $T^t_\Lambda \omega \in \Omega_2$, so

$$\gamma_+ T^t_\Lambda \omega = \lim_{s \to +\infty} T^{-s} T^s_\Lambda T^t_\Lambda \omega$$
$$= \lim_{s \to +\infty} T^t T^{-(s+t)} T^{(s+t)}_\Lambda \omega = T^t \gamma_+ \omega$$

Lemma 6.2. γ_+^{-1} is the metric isomorphism of the systems (Ω_+, Σ, μ) and $(\Omega_2, \Sigma, \mu^\Phi)$.

Proof. It follows from Theorem 5.1 that γ_+^{-1} is a one-to-one transformation of Ω_+ and Ω_2. It is need to prove that for any $A \subset \Omega_+$

$$\mu(A) = \mu^\Phi(\gamma_+^{-1} A) \tag{6.3}$$

Let $B \subset \mathbb{R}^{2\nu}$ be such that there exists $\varepsilon > 0$ such that if $(x, v) \in B$ then $|v| > \varepsilon$. If we shall prove that for any such B

$$\mu(C_{B,K}) = \mu^\Phi(\gamma_+^{-1} C_{B,K})$$

then (6.3) will follow.

Let Σ_Λ be minimal sub-σ-algebra of Σ which contains all $C_{B,K}$ such that the projection of B onto $\mathbb{R}^\nu_x$ does not intersect Λ. Then for any $A \in \Sigma_\Lambda$

$$\mu^\Phi(A) = Z^{-1} \int_\Omega \chi_A(\omega) \exp[-\beta\Phi_\Lambda(\omega)]\, d\mu$$

$$= \int_\Omega \chi_A(\omega)\, d\mu\, Z^{-1} \int_\Omega \exp[-\beta\Phi_\Lambda(\omega)]\, d\mu = \mu(A)$$

because random variable χ_A and $\exp(-\beta\Phi_\Lambda)$ are independent. For any B, which was described above, there exists t_B such that if $(x, v) \in B$ and $t > t_B$ then $x + tv \bar{\in} \Lambda$ and so $T^t C_{B,K} \in \Sigma_\Lambda$. But

$$\mu(C_{B,K}) = \mu(T^t C_{B,K})$$
$$= \mu^\Phi(T^t C_{B,K}) = \mu^\Phi(T_\Lambda^{-t} T^t C_{B,K})$$

if $t > t_B$ then

$$\mu(C_{B,K}) = \mu^\Phi(\gamma_+^{-1} C_{B,K})$$

and γ_+^{-1} is the metric isomorphism. ∎

Theorem 6.1. The dynamical systems $(\Omega, \Sigma, \mu, T^t)$ and $(\Omega, \Sigma, \mu^\Phi, T^t_\Lambda)$ are metrically isomorphic.

Proof. From Lemmas 5.1 and 5.1′ it follows that $\mu(\Omega_+) = 1$ and $\mu^\Phi(\Omega_2) = 1$. The transformation γ_+^{-1} is a metric isomorphism of the dynamical systems $(\Omega, \Sigma, \mu, T^t)$ and $(\Omega, \Sigma, \mu^\Phi, T^t_\Lambda)$.

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